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Certain New Classes of Generalized Closed Sets and Their Applications in Ideal Topological Spaces

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Abstract. In this paper, a type of closed sets, called $*$-g-closed sets, is introduced and studied in an ideal topological space. The class of such sets is found to lie strictly between the class of all closed sets and that of generalized closed sets of Levine [5]. We give some applications of $*$-g-closed set and $*$-g-open set in connection with certain separation axioms.

1. Introduction and Preliminary

As is noticed from the recent literature, there has been a growing trend among some topologists to introduce and study generalized types of closed sets. In 1970, Levine [5] first introduced the novel idea of generalized closed (g-closed, for short) sets, a generalization of closed sets having their own meaningful facets. Moreover g-closed sets were further generalized by Dontchev et al. [2] in an ideal topological space and the said concept of Dontchev et al. has been investigated extensively by Navaneethakrishnan and Joseph [6]. After that many topologists have studied this concept from different angles (for instance see [7], [8], [9] and so on). Following the trend, we have introduced and investigated a kind of generalized closed sets in an ideal topological space. The notion of ideals in general topological spaces is treated in the classic text by Kuratowski [4] and also in [10]. A collection $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an ideal on X if it satisfies the following two conditions:
(i) $A \in \mathcal{I}$ and $A \supseteq B \Rightarrow B \in \mathcal{I}$, and
(ii) $A \in \mathcal{I}$, $B \in \mathcal{I} \Rightarrow A \bigcup B \in \mathcal{I}$.

Let $\mathcal{I}$ be an ideal on a topological space (X, τ) and if $\mathcal{P}(X)$ denotes the set of all subsets of X, a set operator $(.)^* : \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [4] of A with respect to τ and $\mathcal{I}$, was defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \bigcap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. It was also shown in

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[3, 4] that the operator $cl^*(.)$, defined by $cl^*(A) = A \bigcup A^*(\mathcal{I}, \tau)$, is a Kuratowski closure operator giving rise to a topology $\tau^*(\mathcal{I}, \tau)$ on X, called the $*$-topology, finer than τ. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on a space (X, τ), then $(X, \tau, \mathcal{I})$ is called an ideal topological space. The members of τ^* are called τ^*-open or simply $*$-open sets and the complement of a $*$-open set is called a $*$-closed set or equivalently, a subset A of X is called $*$-closed if $A^* \subseteq A$. A subset A of X is said to be g-closed [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X; and the complement of a g-closed subset in X is called a g-open set in X. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_g$-closed [2] if $A^* \subseteq U$ (or equivalently $cl^*(A) \subseteq U$) whenever $A \subseteq U$ and U is open in (X, τ). A subset A of X is said to be $\mathcal{I}_g$-open if $X \setminus A$ is $\mathcal{I}_g$-closed.

In Section 2, we shall introduce a new class of generalized closed sets, termed $*$-g-closed, in an ideal topological space. This class of $*$-g-closed sets lies strictly between the class of closed sets and the class of g-closed sets. We obtain several characterizing properties of $*$-g-closed sets. At the end of the section a precise form of $*$-g-closed set is also incorporated.

In the next section of this paper, we characterize regular and normal spaces in terms of the introduced class of sets. Also we define stronger forms of regularity as well as of normality and investigate their properties.

In the last section of this paper, we continue with certain applications of $*$-closed sets and $\mathcal{I}_g$-closed sets where we define $*$-R_0-space and $\mathcal{I}$-R_0-space respectively, one being stronger and the other weaker than R_0-space. We shall give several characterizing conditions for the introduced types of R_0-spaces.

In what follows in this paper, a space X will always be taken to stand for a topological space (X, τ). For any $A \subseteq X$, int(A) and cl(A) will respectively stand for the interior and closure of A in (X, τ). Whenever we say that a subset A of a space X is open (or closed) it will mean that A is open (or closed) in (X, τ). For open and closed sets with respect to any other topology on X, e.g. τ^*, we would write τ^*-open or simply '$*$-open' and '$*$-closed'.

2. $*$-Generalized Closed Sets

This section is devoted to the introduction and study of a kind of generalized closed sets in an ideal topological space $(X, \tau, \mathcal{I})$, termed $*$-g-closed set. We obtain some characterizations and properties of such sets. Our proposed definition of $*$-g-closed sets goes as follows:

Definition 2.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is said to be a $*$-g-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $*$-open.
The complement of a $*$-g-closed set is called a $*$-g-open set.

Remark 2.2. (*a*) Since every open set is a $*$-open set, it follows that every $*$-g-closed set is g-closed in any ideal topological space $(X, \tau, \mathcal{I})$. But the converse is false as is shown in Example 2.3(*a*).
(*b*) Since $\tau \subseteq \tau^*$, we have every $*$-g-closed set is g-closed in (X, τ^*). At this point it is quite pertinent to raise the question whether the class of g-closed sets in (X, τ^*) is same as that of $*$-g-closed sets in $(X, \tau, \mathcal{I})$. The following Example 2.3(*b*) answers the question in the negative.

Example 2.3. (*a*) Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $\tau^* = \{\phi, \{a\}, \{a, b\}, X\}$. Consider a set $B = \{b\}$. Then it is easy to check that B is g-closed but not $*$-g-closed. Also it is to be noted that the set $C = \{c\}$ is $*$-g-closed but it is not a closed set. Thus we can say that the class of $*$-g-closed sets lie between the class of closed sets and the class of g-closed sets.
(*b*) Consider the ideal topological space $(X, \tau, \mathcal{I})$, where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\phi, \{b\}\}$.

Then $\tau^* = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Let $A = \{b\}$. Then A is g-closed in (X, τ^*) but is not $*$-g-closed in $(X, \tau, \mathcal{I})$. Also it is noted that the $*$-closed set A is not a $*$-g-closed set.

Theorem 2.4. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the union of finite number of $*$-g-closed sets is $*$-g-closed.
Proof. Let A and B be two $*$-g-closed sets in an ideal topological space $(X, \tau, \mathcal{I})$. Suppose that $A \bigcup B \subseteq U$, where U is $*$-open. Then $A \subseteq U$ and $B \subseteq U \Rightarrow cl(A) \subseteq U$ and $cl(B) \subseteq U \Rightarrow cl(A \bigcup B) = cl(A) \bigcup cl(B) \subseteq U \Rightarrow A \bigcup B$ is a $*$-g-closed set.

Example 2.5. The intersection of two $*$-g-closed sets is not necessarily a $*$-g-closed set. In fact, let $X = \{a, b, c\}$, $\tau = \{\phi, \{b\}, X\}$ and $\mathcal{I} = \{\phi\}$. Let $A = \{a, b\}$ and $B = \{b, c\}$. Then A and B are $*$-g-closed, but $A \bigcap B = \{b\}$ is not $*$-g-closed.

The following result gives some characterizing conditions of $*$-g-closed sets:

Theorem 2.6. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following are equivalent:
(a) A is a $*$-g-closed set.
(b) For each $x \in cl(A)$, $cl^*(\{x\}) \bigcap A \neq \phi$.
(c) $cl(A) \setminus A$ contains no nonempty $*$-closed sets.
Proof. (a)⇒(b): Suppose that $x \in cl(A)$. If possible, let $cl^*(\{x\}) \bigcap A = \phi$. Then $A \subseteq X \setminus cl^*(\{x\}) \Rightarrow cl(A) \subseteq X \setminus cl^*(\{x\})$ (by (a)) $\Rightarrow cl(A) \bigcap cl^*(\{x\}) = \phi$, a contradiction.
(b)⇒ (c): Suppose that $F \subseteq cl(A) \setminus A$, where F is nonempty and $*$-closed. If $x \in F$, then $x \in cl(A)$ and so by (b), $\phi \neq cl^*(\{x\}) \bigcap A \subseteq F \bigcap A \subseteq (cl(A) \setminus A) \bigcap A$, a contradiction. Thus $F = \phi$.
(c) ⇒ (a): Suppose that $A \subseteq U$, where U is $*$-open. Then $cl(A) \bigcap (X \setminus U) \subseteq cl(A) \bigcap (X \setminus A) = cl(A) \setminus A$. Since $cl(A) \bigcap (X \setminus U)$ is $*$-closed, it follows by (c) that $cl(A) \bigcap (X \setminus U) = \phi$. Thus $cl(A) \subseteq U$ and hence A is a $*$-g-closed set.

Theorem 2.7. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Let A and B be subsets of X such that $A \subseteq B \subseteq cl(A)$ and A is a $*$-g-closed set. Then B is also $*$-g-closed.
Proof. Let $B \subseteq U$, where U is $*$-open. Then $A \subseteq U$ and A is $*$-g-closed $\Rightarrow cl(A) \subseteq U \Rightarrow cl(B) \subseteq U \Rightarrow B$ is a $*$-g-closed set.

Remark 2.8. It may so happen that that two subsets A, B of a space X are $*$-g-closed and $A \subseteq B$, but $B \nsubseteq cl(A)$. In fact, let $X = \{a, b, c\}$, $\tau = \{\phi, \{c\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. Let $A = \{a\}$ and $B = \{a, c\}$. Then A and B are $*$-g-closed sets, but $A \subseteq B \nsubseteq cl(A)$.

Theorem 2.9. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A(\subseteq X)$ be a $*$-g-closed set. Then A is closed if and only if $cl(A) \setminus A$ is $*$-closed.
Proof. Let A be closed, then $cl(A) \setminus A = \phi$ and so $cl(A) \setminus A$ is $*$-closed.
Conversely, suppose that $cl(A) \setminus A$ is $*$-closed. Since A is $*$-g-closed, by Theorem 2.6, $cl(A) \setminus A = \phi$ and so A is closed.

Theorem 2.10. For an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent:
(a) Every $*$-open set is closed.
(b) Every subset of X is $*$-g-closed.
Proof. (a) ⇒ (b): Let $A \subseteq X$ and $A \subseteq U$ where U is $*$-open. Then by (a), U is closed and so $cl(A) \subseteq cl(U) = U$. Thus A is $*$-g-closed.
(b)⇒ (a): Let U be a $*$-open set. Then by (b), U is $*$-g-closed and hence $cl(U) \subseteq U \Rightarrow U$ is closed.

Corollary 2.11. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is $*$-open and $*$-g-closed then A is closed.

Theorem 2.12. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $*$-g-open if and only if

$F \subseteq int(A)$ whenever $F \subseteq A$ and F is a $*$-closed set.
Proof. Suppose that A is $*$-g-open and $F \subseteq A$, where F is $*$-closed. Then $X \setminus A \subseteq X \setminus F$. Since $X \setminus A$ is $*$-g-closed, $cl(X \setminus A) \subseteq X \setminus F \Rightarrow F \subseteq int(A)$.
Conversely, let the condition hold. Let $X \setminus A \subseteq U$, where U is $*$-open. Then $X \setminus U \subseteq A$ where $X \setminus U$ is $*$-closed and so by hypothesis, $X \setminus U \subseteq int(A) \Rightarrow cl(X \setminus A) \subseteq U \Rightarrow X \setminus A$ is a $*$-g-closed set $\Rightarrow A$ is a $*$-g-open set.

Theorem 2.13. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X such that $int(A) \subseteq B \subseteq A$. If A is $*$-g-open then B is also $*$-g-open.
Proof. Follows from Theorem 2.7.

Theorem 2.14. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $*$-g-open if and only if $U = X$ whenever $int(A) \bigcup (X \setminus A) \subseteq U$ and U is $*$-open.
Proof. Let A be $*$-g-open and $int(A) \bigcup (X \setminus A) \subseteq U$, where U is $*$-open. Then $X \setminus U \subseteq X \setminus (int(A) \bigcup (X \setminus A)) \Rightarrow X \setminus U \subseteq cl(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $*$-g-closed, by Theorem 2.6, it follows that $X \setminus U = \phi \Rightarrow U = X$.
Conversely, suppose that $F \subseteq A$, where F is $*$-closed. Then $int(A) \bigcup (X \setminus A) \subseteq int(A) \bigcup (X \setminus F)$. Therefore by hypothesis, $int(A) \bigcup (X \setminus F) = X \Rightarrow F \subseteq int(A) \Rightarrow A$ is $*$-g-open (by Theorem 2.12).

Theorem 2.15. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $*$-g-closed if and only if $cl(A) \setminus A$ is $*$-g-open.
Proof. Let A be $*$-g-closed. Let $F \subseteq cl(A) \setminus A$, where F is $*$-closed. Then by Theorem 2.6, $F = \phi$. Thus $F \subseteq int(cl(A) \setminus A)$ and hence by Theorem 2.12, $cl(A) \setminus A$ is $*$-g-open.
Conversely, let $A \subseteq U$, where U is $*$-open. Then $cl(A) \bigcap (X \setminus U) \subseteq cl(A) \bigcap (X \setminus A) = cl(A) \setminus A$. Since $cl(A) \bigcap (X \setminus U)$ is $*$-closed and $cl(A) \setminus A$ is $*$-g-open, by Theorem 2.12 we have, $cl(A) \bigcap (X \setminus U) \subseteq int(cl(A) \setminus A) = \phi$. Thus $cl(A) \subseteq U$. Hence A is $*$-g-closed.

Theorem 2.16. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for each $x \in X$, either $\{x\}$ is $*$-closed or $*$-g-open.
Proof. Let $x \in X$ be such that $\{x\}$ is not $*$-closed. Then X is the only $*$-open set containing $X \setminus \{x\} \Rightarrow X \setminus \{x\}$ is $*$-g-closed $\Rightarrow \{x\}$ is $*$-g-open.

Theorem 2.17. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is $*$-g-closed if and only if $A = F \setminus N$ where F is closed and N contains no nonempty $*$-closed set.
Proof. Let A be a $*$-g-closed set. Then by Theorem 2.6, $cl(A) \setminus A = N(say)$ contains no nonempty $*$-closed set. Let $F = cl(A)$. Therefore $F \setminus N = cl(A) \setminus (cl(A) \setminus A) = A$.
Conversely, let $A = F \setminus N$, where F is closed and N contains no nonempty $*$-closed set. Let $A \subseteq U$, where U is $*$-open. Thus $F \setminus N \subseteq U \Rightarrow F \bigcap (X \setminus U) \subseteq N$. Since $F \bigcap (X \setminus U)$ is a $*$-closed set, $F \bigcap (X \setminus U) = \phi$ and so $F \subseteq U$. Now $A \subseteq F \Rightarrow cl(A) \subseteq F \subseteq U \Rightarrow A$ is $*$-g-closed.

3. $*$-g-Regular and $*$-g-normal spaces

In this section, we shall introduce and investigate two kinds of separation axioms, termed $*$-g-regularity and $*$-g-normality, stronger than regularity and normality respectively. Before that we prove two results providing characterizations of regularity and normality of a topological space in terms of $*$-g-open sets.

Theorem 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent:
(a) (X, τ) is regular.
(b) For each closed set F and each $x \notin F$, there exist an open set U and a $*$-g-open set V such that $x \in U$, $F \subseteq V$ and $U \bigcap V = \phi$.
(c) For each $A \subseteq X$ and each closed set F in X with $A \bigcap F = \phi$, there exist an open set U and a $*$-g-open set V such that $A \bigcap U \neq \phi$, $F \subseteq V$ and $U \bigcap V = \phi$.

Proof. (a) ⇒ (b): Follows from the fact that every open set is a $*$-g-open set.
(b) ⇒ (c): Let $A \subseteq X$ and F be a closed set in X such that $A \bigcap F = \phi$. Then by (b), for each $x \in A$, there exist an open set U and a $*$-g-open set V such that $x \in U$, $F \subseteq V$ and $U \bigcap V = \phi$. Thus $A \bigcap U \neq \phi$, $F \subseteq V$ and $U \bigcap V = \phi$.
(c) ⇒ (a): Let F be a closed set and $x \notin F$. Then by (c), there exist an open set U and a $*$-g-open set V such that $x \in U$, $F \subseteq V$ and $U \bigcap V = \phi$. Since $F \subseteq V$ and V is $*$-g-open, by Theorem 2.12, $F \subseteq int(V) = W(say)$. Thus $x \in U$, $F \subseteq W$, where U and W are disjoint open sets. Hence (X, τ) is regular.

Theorem 3.2. For an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent:
(a) (X, τ) is normal.
(b) For each pair of disjoint closed sets F and K, there exist disjoint $*$-g-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.
(c) For each closed set F and each open set V containing F, there exists a $*$-g-open set U such that $F \subseteq U \subseteq cl(U) \subseteq V$.
(d) For each closed set F and each $*$-g-open set V containing F, there exists a $*$-g-open set U such that $F \subseteq U \subseteq cl(U) \subseteq int(V)$.
(e) For each $*$-g-closed set F and each open set V containing F, there exists a $*$-g-open set U such that $F \subseteq cl(F) \subseteq U \subseteq cl(U) \subseteq V$.
Proof. (a) ⇒ (b): Follows from the fact that every open set is a $*$-g-open set.
(b)⇒ (c): Let F be a closed set such that $F \subseteq V$, where V is open. Then F and $X \setminus V$ are disjoint closed sets and by (b), there exist disjoint $*$-g-open sets U and W such that $F \subseteq U$ and $X \setminus V \subseteq W$. Now W is $*$-g-open and $X \setminus V \subseteq W \Rightarrow X \setminus V \subseteq int(W)$ (by Theorem 2.12). Thus $F \subseteq U \subseteq cl(U) \subseteq cl(X \setminus W) \subseteq V$.
(c) ⇒ (d): Let F be closed such that $F \subseteq V$, where V is $*$-g-open. Then by Theorem 2.12, $F \subseteq int(V)$ and hence by (c), there exists a $*$-g-open set U such that $F \subseteq U \subseteq cl(U) \subseteq int(V)$.
(d) ⇒ (e): Let F be a $*$-g-closed set such that $F \subseteq V$, where V is open. Then $cl(F) \subseteq V$ and by (d), there exists a $*$-g-open set U such that $F \subseteq U \subseteq cl(U) \subseteq int(V) = V$.
(e) ⇒ (a): Let F and K be any two disjoint closed sets in X. Then $F \subseteq X \setminus K$ where F is $*$-g-closed and so by (e), there exists a $*$-g-open set U such that $F \subseteq U \subseteq cl(U) \subseteq (X \setminus K)$. Since F is $*$-closed and $F \subseteq U$ where U is $*$-g-open, by Theorem 2.12 we have, $F \subseteq int(U)$. Put $G = int(U)$ and $H = X \setminus cl(U)$. Then G and H are disjoint open sets such that $F \subseteq G$ and $K \subseteq H$ and hence X is normal.

We now define a stronger form of regularity as follows:

Definition 3.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then X is said to be $*$-g-regular if for each $*$-g-closed set F and each $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

Remark 3.4. Since every closed set is $*$-g-closed, it follows that every $*$-g-regular space is regular. But the converse is false as is shown by the following example.

Example 3.5. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $\tau^* = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then (X, τ) is regular but is not $*$-g-regular. In fact, $F = \{c\}$ is $*$-g-closed and $b \notin F$ but there are no disjoint open sets containing b and F.

The following result gives some equivalent conditions of $*$-g-regularity:

Theorem 3.6. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent:
(a) (X, τ) is $*$-g-regular.
(b) For each $x \in X$ and each $*$-g-open set U containing x, there exists an open set V in X such that $x \in V \subseteq \mathrm{cl}(V) \subseteq U$.
(c) For each $x \in X$ and each $*$-g-closed set F with $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $cl(F) \subseteq V$.
(d) For each $*$-g-closed set F and each point $x \in X \setminus F$, there exist open sets U and V of X such that $x \in U$,

$F \subseteq V$ and $cl(U) \bigcap cl(V) = \phi$.

Proof. (a)⇒ (b): For a given $x \in X$, and any $*$-g-open set U containing x, there exist disjoint open sets V and W such that $x \in V$ and $X \setminus U \subseteq W$. Then $\text{cl}(V) \subseteq U$. Thus $x \in V \subseteq \text{cl}(V) \subseteq U$.

(b)⇒ (a): Let $x \in X$ and F be a $*$-g-closed set with $x \notin F$. Then by (b), there exists an open set V such that $x \in V \subseteq \text{cl}(V) \subseteq X \setminus F$. Put $W = X \setminus \text{cl}(V)$ and then V and W strongly separate x and F.

(a) ⇒ (c): For $x \in X$ and a $*$-g-closed set F not containing x, there exist disjoint open sets U and V in X such that $x \in U$ and $F \subseteq V$. Since F is $*$-g-closed, $\text{cl}(F) \subseteq V$.

(c) ⇒ (d): Let $x \in X$ and F be a $*$-g-closed set not containing x. Then by (c), there exist disjoint open sets W and V such that $x \in W$ and $\text{cl}(F) \subseteq V$. Now $\text{cl}(V)$ is $*$-g-closed and $x \notin \text{cl}(V)$. Then again by (c), there exist open sets G and H in X such that $x \in G$, $cl(cl(V)) = cl(V) \subseteq H$ and $G \bigcap H = \phi$. Now put $U = W \bigcap G$, then U and V are open subsets of X such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \bigcap \text{cl}(V) = \phi$.

(d)⇒ (a): It is clear.

We now define another type of separation axiom in an ideal topological space as follows:

Definition 3.7. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then (X, τ) is said to be $*$-g-normal if for each pair of disjoint $*$-g-closed sets F and K, there exist disjoint open sets U and V in X such that $F \subseteq U$ and $K \subseteq V$.

Remark 3.8. Since every closed set is $*$-g-closed for any ideal topological space $(X, \tau, \mathcal{I})$, every $*$-g-normal space is normal. But the converse is false as is shown by the following example.

Example 3.9. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, X\}$ and $\mathcal{I} = \{\phi\}$. Then (X, τ) is normal but not $*$-g-normal. In fact, $F = \{a, b\}$ and $K = \{c, d\}$ are disjoint $*$-g-closed sets but they cannot be separated by disjoint open sets in X.

Theorem 3.10. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent:
(a) X is $*$-g-normal.
(b) For each $*$-g-closed set F and any $*$-g-open set U containing F, there exists an open set V of X such that $F \subseteq V \subseteq \text{cl}(V) \subseteq U$.
(c) For each pair of disjoint $*$-g-closed sets F and K, there exists an open set U in X containing F such that $\text{cl}(U) \bigcap K = \phi$.
(d) For each pair of disjoint $*$-g-closed sets F and K, there exist open sets U and V in X such that $F \subseteq U$, $K \subseteq V$ and $\text{cl}(U) \bigcap \text{cl}(V) = \phi$.

Proof. We only prove '(a) ⇒ (b)'; other implications viz. '$(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$' can be proved following usual pattern. Let F be a $*$-g-closed set and U a $*$-g-open set such that $F \subseteq U$. Then by (a), there exist disjoint open sets V and W such that $F \subseteq V$ and $X \setminus U \subseteq W$. Thus $F \subseteq V \subseteq cl(V) \subseteq (X \setminus W) \subseteq U$.

Theorem 3.11. Let $(X, \tau, \mathcal{I})$ be a $*$-g-normal ideal topological space. If F is a $*$-g-closed set and V be a $*$-g-open set in X such that $F \subseteq V$, then there exists an open set U in X such that $F \subseteq cl(F) \subseteq U \subseteq int(V) \subseteq V$.

Proof. Let F be $*$-g-closed and V be a $*$-g-open set in X such that $F \subseteq V$. Then F and $X \setminus V$ are disjoint $*$-g-closed sets and so by $*$-g-normality of X, there exist disjoint open sets U and W such that $F \subseteq U$ and $X \setminus V \subseteq W$. Since $F \subseteq U$ and F is $*$-g-closed, $cl(F) \subseteq U$. Similarly $cl(X \setminus V) \subseteq W$. Now $U \subseteq X \setminus W \subseteq int(V)$. Thus $F \subseteq cl(F) \subseteq U \subseteq int(V) \subseteq V$.

4. $*$-R_0-Space and $\mathcal{I}$-R_0-space

The notion of R_0-space, first introduced by Davis [1], has been studied by many topologists. In this section, we introduce two kinds of separation axioms, one being stronger and another weaker than R_0-spaces, and obtain some characterizations of these spaces. We begin with the following definition recalled from [1].

Definition 4.1. A topological space (X, τ) is said to be an R_0-space if every open set contains the closure of each of its singletons.

We now define as follows for an ideal topological space:

Definition 4.2. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be
(i) a $*$-R_0-space if for every $*$-open set U and each $x \in U$, one has $cl(\{x\}) \subseteq U$ i.e., every singleton is $*$-g-closed.
(ii) an $\mathcal{I}$-R_0-space if ($x \in U$, where U is open in $X \Rightarrow (\{x\})^* \subseteq U$) i.e., every singleton is $\mathcal{I}_g$-closed.

Remark 4.3. (a) Since every open set is $*$-open, it follows that every $*$-R_0-space is an R_0-space. But the converse is false as is shown by the following Example 4.4.
(b) Since every g-closed set is $\mathcal{I}_g$-closed, it follows that every R_0-space is an $\mathcal{I}$-R_0-space. But the converse is false. This is shown by Example 4.5.

Example 4.4. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\phi, \{a\}, \{a, c\}\}$. Then (X, τ) is an R_0-space but $(X, \tau, \mathcal{I})$ is not a $*$-R_0-space.

Example 4.5. Consider an ideal topological space $(X, \tau, \mathcal{I})$, where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\mathcal{I} = \{\phi, \{a\}\}$. Then (X, τ) is not an R_0-space as $\{a\}$ is not g-closed, although $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-R_0-space.

The following result gives some equivalent conditions of $*$-R_0-property.

Theorem 4.6. For an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent:
(a) $(X, \tau, \mathcal{I})$ is a $*$-R_0-space.
(b) F is a $*$-closed set with $x \notin F \Rightarrow F \subseteq U$ and $x \notin U$ for some $U \in \tau$.
(c) F is a $*$-closed set with $x \notin F \Rightarrow F \bigcap cl(\{x\}) = \phi$.
(d) For any two distinct points x and y of X, $x \notin cl^*(\{y\}) \Rightarrow cl(\{x\}) \bigcap cl^*(\{y\}) = \phi$.
Proof. (a) ⇒ (b): Let F be a $*$-closed set and $x \notin F$. Then $x \in X \setminus F$ and so by (a), $cl(\{x\}) \subseteq X \setminus F \Rightarrow F \subseteq X \setminus cl(\{x\})$. Put $X \setminus cl(\{x\}) = U$. Then U is an open set in X such that $F \subseteq U$ and $x \notin U$.
(b) ⇒ (c): Let F be a $*$-closed set and $x \notin F$. Then by (b), there exists an open set U in X such that $F \subseteq U$ and $x \notin U$. Thus $U \bigcap cl(\{x\}) = \phi$ and hence $F \bigcap cl(\{x\}) = \phi$.
(c) ⇒ (d): It is clear.
(d) ⇒ (a): Let $x \in U$, where U is $*$-open. Then for each $y \notin U$, $x \notin cl^*(\{y\})$ and hence by (d), $cl(\{x\} \bigcap cl^*(\{y\}) = \phi$, for each $y \notin U \Rightarrow cl(\{x\} \bigcap [\bigcup \{cl^*(\{y\}) : y \in X \setminus U\}] = \phi$. Now U is $*$-open and $y \in X \setminus U \Rightarrow \{y\} \subseteq cl^*(\{y\}) \subseteq cl^*(X \setminus U) = X \setminus U$. Thus $X \setminus U = \bigcup \{cl^*(\{y\}) : y \in X \setminus U\}$. Therefore $cl(\{x\}) \bigcap (X \setminus U) = \phi$ i.e., $cl(\{x\}) \subseteq U$. Hence $(X, \tau, \mathcal{I})$ is a $*$-R_0-space.

The following result gives some more characterizations of $*$-R_0-spaces:

Theorem 4.7. For an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent:
(a) $(X, \tau, \mathcal{I})$ is a $*$-R_0-space.
(b) For each $(\phi \neq) A \subseteq X$ and $*$-open set U with $A \bigcap U \neq \phi$, there exists a closed set F such that $A \bigcap F \neq \phi$ and $F \subseteq U$.
(c) For each $*$-open set U, $U = \bigcup \{F : F$ is closed and $F \subseteq U\}$.
(d) For each $*$-closed set F, $F = \bigcap \{U : U$ is open and $F \subseteq U\}$.
Proof. (a) ⇒ (b) : Let $A (\subseteq X)$ be such that $A \bigcap U \neq \phi$ where U is $*$-open. Let $x \in A \bigcap U$. Now, $x \in U \Rightarrow cl(\{x\}) \subseteq U$. Let $cl(\{x\}) = F$. Then F is a closed set with $F \subseteq U$ and $A \bigcap F \neq \phi$.
(b) ⇒ (c) : Let U be $*$-open. Now $\bigcup \{F : F$ is closed and $F \subseteq U\} \subseteq U$. Let $x \in U$. Then by (b) there exists a closed set F containing x and $F \subseteq U$. Thus $x \in F \subseteq \bigcup \{K : K$ is closed and $K \subseteq U\}$. Therefore for each $*$-open set U, $U = \bigcup \{F : F$ is closed and $F \subseteq U\}$.
(c) ⇒ (d) : It is clear.
(d) ⇒ (a) : Let U be a $*$-open set and let $x \in U$. We need to show that $\mathrm{cl}\{x\} \subseteq U$. If not, then there exists an

$y \in cl\{x\}$ such that $y \notin U$. As U is a $*$-open neighbourhood of each of its points, it follows that $cl^*\{y\} \bigcap U = \phi$. Now, $cl^*\{y\}$ is $*$-closed set and hence by (d), $cl^*\{y\} = \bigcap\{V : V$ is open in X and $cl^*\{y\} \subseteq V\}$ so that $(\bigcap\{V : V$ is open in X and $cl^*\{y\} \subseteq V\}) \bigcap U = \phi$ i.e., $x \notin \bigcap\{V : V$ is open in X and $cl^*\{y\} \subseteq V\} \Rightarrow$ there exists an open set V in X such that $x \notin V$ and $cl^*\{y\} \subseteq V$. As V is open set containing y such that $x \notin V$, we have $y \notin cl\{x\}$, a contradiction.

Theorem 4.8. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent:
(a) $(X, \tau, \mathcal{I})$ is a $*$-R_0-space.
(b) For any two distinct points x and y, $x \in cl^*(\{y\})$ if and only if $y \in cl(\{x\})$.
Proof. (a) $\Rightarrow$ (b) : Let $x \in cl^*(\{y\})$ and U be an open set containing y. Then using (a), we have $cl(\{y\}) \subseteq U$. Thus $cl^*(\{y\}) \subseteq cl(\{y\}) \subseteq U \Rightarrow x \in U \Rightarrow y \in cl(\{x\})$. Again, let $y \in cl(\{x\})$ and U be any $*$-open set containing x. Then by (a), we have $cl(\{x\}) \subseteq U \Rightarrow y \in U \Rightarrow x \in cl^*(\{y\})$.
(b) $\Rightarrow$ (a) : Let U be any $*$-open set and $x \in U$. If $(x \neq) y \notin U$ then $x \notin cl^*(\{y\})$ and hence $y \notin cl(\{x\})$. This shows that $cl(\{x\}) \subseteq U$. Hence $(X, \tau, \mathcal{I})$ is a $*$-R_0-space.

The following theorem gives some characterizations of $\mathcal{I}$-R_0-spaces:

Theorem 4.9. For an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent:
(a) $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-R_0-space.
(b) For each $x \in U$, where U is open in X, $cl^*(\{x\}) \subseteq U$.
(c) For each closed set F with $x \notin F$, $F \subseteq U$ and $x \notin U$ for some $*$-open set U.
(d) For each closed set F with $x \notin F$, $F \bigcap cl^*(\{x\}) = \phi$.
(e) For any two distinct points x and y of X, $x \notin cl(\{y\}) \Rightarrow cl^*(\{x\}) \bigcap cl(\{y\}) = \phi$.
Proof. (a) $\Rightarrow$ (b): Let U be open in X and $x \in U$. Then by (a), $(\{x\})^* \subseteq U$. Thus $cl^*(\{x\}) = \{x\} \bigcup (\{x\})^* \subseteq U$.
(b) $\Rightarrow$ (c): Let F be closed and $x \notin F$. Then $x \in X \setminus F$ and so by (b), $cl^*(\{x\}) \subseteq X \setminus F$ i.e., $F \subseteq X \setminus cl^*(\{x\})$. Put $U = X \setminus cl^*(\{x\})$. Then U is $*$-open such that $F \subseteq U$ and $x \notin U$.
(c) $\Rightarrow$ (d): Let F closed in X and $x \notin F$. Then by (c), there exists a $*$-open set U such that $F \subseteq U$ and $x \notin U$, and so $U \bigcap cl^*(\{x\}) = \phi$. Consequently, $F \bigcap cl^*(\{x\}) = \phi$.
(d) $\Rightarrow$ (e): It is clear.
(e) $\Rightarrow$ (a): Let U be open in X and $x \in U$. Then for each $y \in X \setminus U$, $x \notin cl(\{y\})$. Therefore by (e), $cl^*(\{x\}) \bigcap cl(\{y\}) = \phi$ for each $y \in X \setminus U \Rightarrow cl^*(\{x\}) \bigcap [\bigcup \{cl(\{y\}) : y \in X \setminus U] = \phi \Rightarrow cl^*(\{x\}) \bigcap (X \setminus U) = \phi \Rightarrow cl^*(\{x\}) \subseteq U$. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-R_0-space.

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